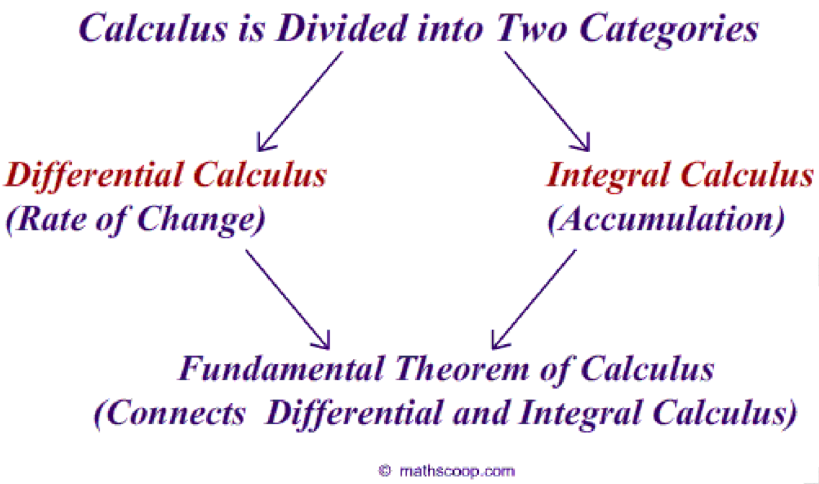
TOPIC 1: FUNCTIONS

1.1 <u>What is Calculus</u>?

"Calculus is the study of how things change. It provides a framework for modeling systems in which there is change, and a way to deduce the predictions of such models."
(Calculus for Beginners and Artists by Daniel Kleitman)



Go to YouTube to view the video(s) "What is calculus?"

1.2 Numbers and Intervals

Real numbers

Natural numbers (N) : 1, 2,3,4, …

Integers (I or Z) : . . . , −4, −3, −2, −1, 0, 1, 2, 3, 4, . . .

Natural numbers are integers too.

Rational numbers (Q) : Any number that can be written as an integer divided by a non-zero integer. Integers are rational numbers too.

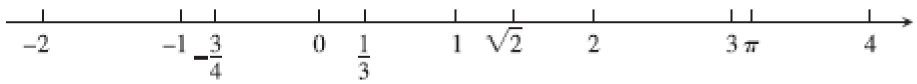
Some examples of rational numbers are: $\frac{3}{4}, \frac{14}{6}, 32 = \frac{32}{1}, 0.23 = \frac{23}{100}, -\frac{5}{3} = \frac{-5}{3} = \frac{5}{-3}$

Real numbers (R):

Some examples of **irrational** numbers are: $\sqrt{2}, \sqrt{15}, 1+\sqrt{3}, \sqrt[3]{10}, \pi, e, \sin 15°$

The rational and irrational numbers together comprise what is called the ***real number system***. The rational numbers and irrational numbers are all real numbers.

The real numbers can be represented by points on a line.



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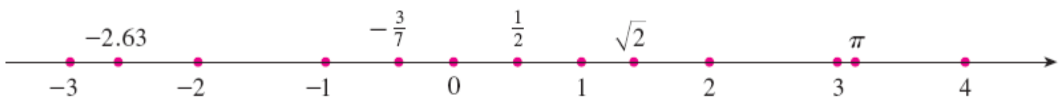

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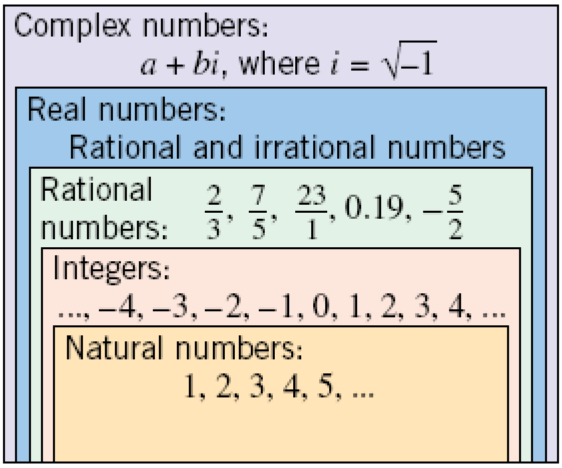
Complex numbers

Numbers of the form $a+bi$ where $i=\sqrt{-1}$.

Note that every real number a is also a complex number because it can be written as
$a = a + 0i$.



Intervals

Notation	Set description	Picture
(a, b)	$\{x \mid a < x < b\}$	a b
$[a, b]$	$\{x \mid a \leq x \leq b\}$	a b
$[a, b)$	$\{x \mid a \leq x < b\}$	a b
$(a, b]$	$\{x \mid a < x \leq b\}$	a b
(a, ∞)	$\{x \mid x > a\}$	a
$[a, \infty)$	$\{x \mid x \geq a\}$	a
$(-\infty, b)$	$\{x \mid x < b\}$	b
$(-\infty, b]$	$\{x \mid x \leq b\}$	b
$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	

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Here a and b are real numbers with $a < b$.

[Some **terms**: finite interval, infinite interval, endpoints (boundary points), open, closed, half-open.]

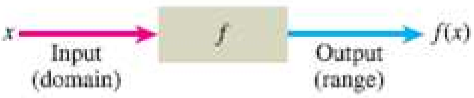
Although the symbol ∞ ("infinity") is used in some of the notations, this does not mean that ∞ is a number. ∞ is NOT a real number.
For example, the notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

1.3 <u>Functions</u>

When the value of one variable quantity, say *y*, depends on the value of another variable quantity, which we might call *x*. We say that "*y* is a function of *x*" and write this symbolically as

$$y = f(x) \quad (\text{"}y \text{ equals } f \text{ of } x\text{"}).$$

In this notation, the symbol *f* represents the function, the letter *x* is the **independent variable** representing the input value of *f*, and *y* is the **dependent variable** or output value of $f(x)$.



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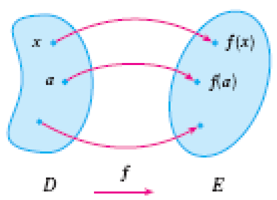
Seeing function as a machine.

> **DEFINITION.** A **function** f from a set D to a set E is a rule that assigns to each element $x \in D$ a unique (exactly one) element $f(x) \in E$.

The element $f(x)$ is called the **value of f at x**, and is read "*f* of *x*."



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We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the **domain of *f*** while the set E is the **codomain of *f*** ; the **range of *f*** is the set of all possible values of $f(x)$ as x varies throughout the domain.

The **graph of *f*** is the set of ordered pairs $\{(x, f(x)) \mid x \in D\}$

(Notice that these are input-output pairs.) In other words, the graph of consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and *x* is in the domain of ***f***.

Four possible ways of representing a function:

- Verbally (Describe in words)
- Numerically (Use a table of values)
- Graphically/Visually (Use a graph)
- Algebraically (Use formula(s) or algebraic expression(s))

To specify a function f you must

(1) give a rule which tells you how to compute the value $f(x)$ of the function for a given real number *x*, and

(2) say for which real numbers *x* the rule may be applied.

Examples:

(i) $f(x) = x^2$

Values of $f(-2), f(0), f(3)$?
What is the domain of f ?
Graph of f ?

(ii) $g(x) = \sqrt{x}$

Values of $g(-2), g(0), g(3)$?
What is the domain of g ?
Graph of g ?

When we define a function $y = f(x)$ with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values, the so-called **natural domain**.

(iii) $h(x) = x^2$, for $x \in [-2,2]$

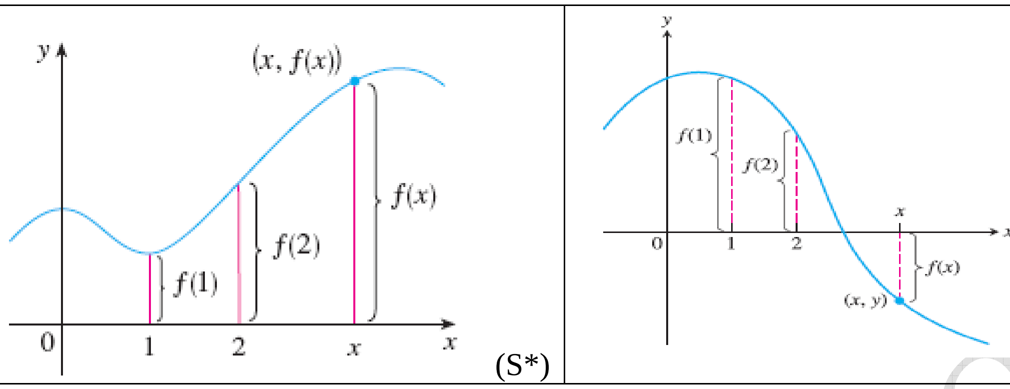
Values of $h(-2), h(0), h(3)$?
What is the domain of h?
Graph of h ?

Find the domain of each function. Write the domain in the form of an interval or union of intervals.

$f(x) = \frac{1}{x}$	
$g(x) = \sqrt{9-x}$	
$h(x) = \sqrt{4-x^2}$	
$m(x) = \sqrt{x+3}$	
$n(x) = \frac{1}{x^2 - 2x}$	
$p(x) = \frac{x}{\lvert x \rvert}$	

Graphing a function. You get the graph of a function f by drawing all points whose coordinates are (x, y) where x must be in the domain of f and $y = f(x)$.

Graph of $f = \{(x, f(x)) \mid x \in D\}$
It consists of all points in the coordinate plane such that x is in the domain of f and $y = f(x)$.



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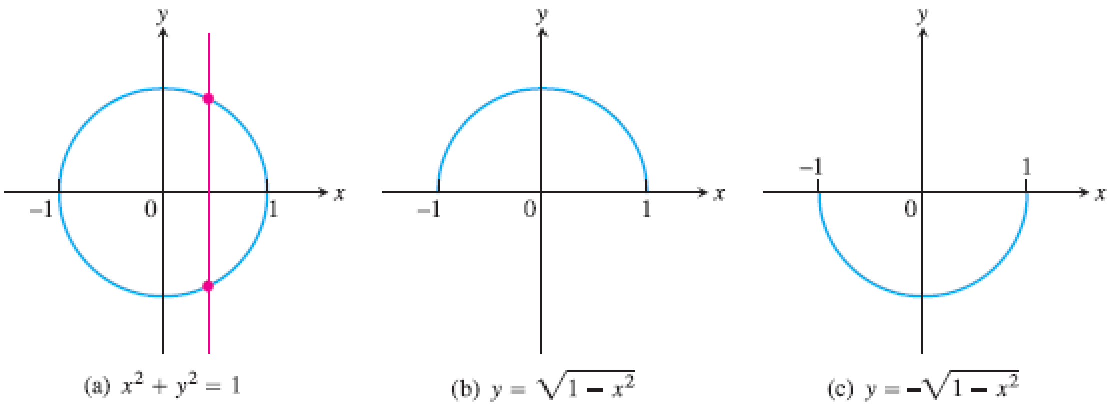
The Vertical Line Test

Given a curve in the *xy*-plane, it is the graph of a function of *x* if and only if no vertical line intersects the curve more than once.

Examples:



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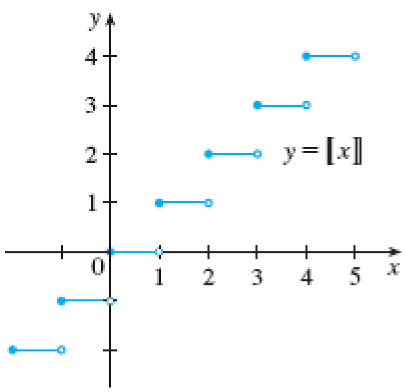
Graph of the **greatest integer function** or the **integer floor function**



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Piecewise Defined Functions

DEFINITION. A **piecewise defined function** is a function which is defined symbolically using two or more formulas.

Examples: Sketch the graph of each function

$$f(x)=\begin{cases} x^2 & \text{if } x<2 \\ 1+x & \text{if } x\geq 2 \end{cases}$$

$$g(x)=\begin{cases} x & \text{if } 0\leq x\leq 2 \\ 4-x & \text{if } 2<x\leq 4 \\ 0 & \text{if } x>4 \end{cases}$$

$$h(x)=\begin{cases} x & \text{if } 0\leq x\leq 2 \\ 4-x & \text{if } 2<x\leq 4 \\ 0 & \text{if } x\geq 4 \end{cases}$$ (Is this a function?)

$$h(x)=\begin{cases} x & \text{if } 0\leq x\leq 2 \\ 4-x & \text{if } 2<x\leq 4 \\ 2 & \text{if } x\geq 4 \end{cases}$$ [Explain why this is NOT a function?]

Even Functions and Odd Functions: Symmetry

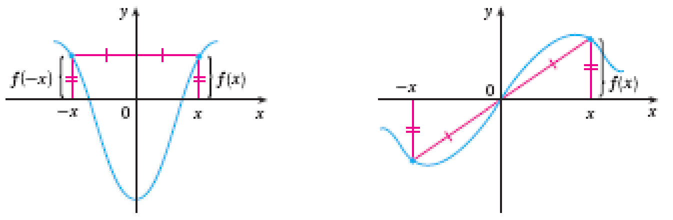
Definitions

A function $y=f(x)$ is an

even function of x if $f(-x)=f(x)$

odd function of x if $f(-x)=-f(x)$

for every x in the domain of f.



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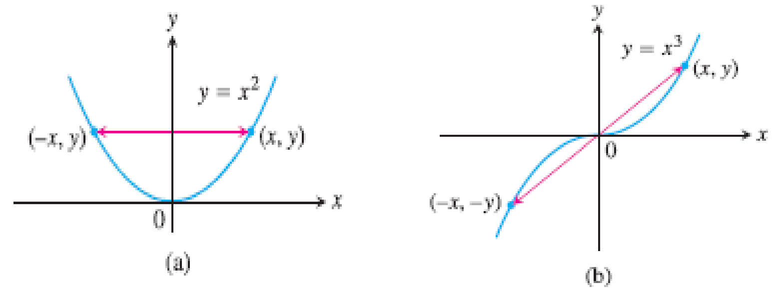

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The graph of an **even** function is **symmetric about the y-axis**.
The graph of an **odd** function is **symmetric about the origin**.

Examples

(i) $f(x)=1-x^2$: $\quad f(-x)=1-(-x)^2=1-x^2=f(x)$. Therefore f is an even function.

(ii) $g(x)=x^3+x$: $\quad g(-x)=(-x)^3+(-x)=-x^3-x=-(x^3+x)=-g(x)$

Conclusion?

(iii) $h(x)=x-x^2$: $\quad h(-x)=(-x)-(-x)^2=-x-x^2$

$$-h(x)=-(x-x^2)=-x+x^2$$

Since $h(x)\neq h(-x)$, h is not an even function.
Since $h(-x)\neq -h(-x)$, h is not an odd function.
We conclude that h is neither even nor odd.

Increasing and Decreasing Functions

Definitions

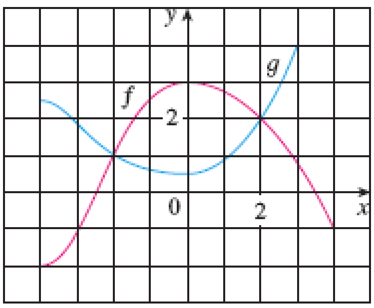
A function f is said to be **increasing on an interval** I if

$$f(x_1)<f(x_2) \quad \text{whenever } x_1<x_2 \text{ in } I.$$

A function f is said to be **decreasing on an interval** I if

$$f(x_1)>f(x_2) \quad \text{whenever } x_1<x_2 \text{ in } I.$$

Example



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Given the graphs of f and g,
(i) on what interval is f increasing?
(ii) on what interval is g decreasing?

Common functions:

Constant functions
Linear functions
Power functions
Polynomials
Rational functions
Algebraic functions - Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots)
Trigonometric functions
Exponential functions
Logarithmic functions

Combining functions

Functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

If f and g are functions, we define
functions $f+g$, $f-g$ and fg by the formulas
$(f+g)(x) = f(x)+g(x)$
$(f-g)(x) = f(x)-g(x)$
$(fg)(x) = f(x)g(x)$ for $x \in D(f) \cap D(g)$.

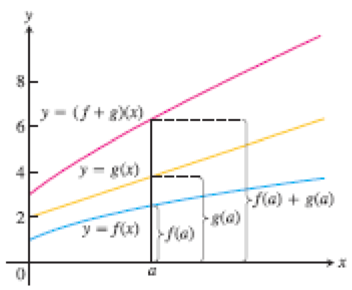
Notice that the + sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the + on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

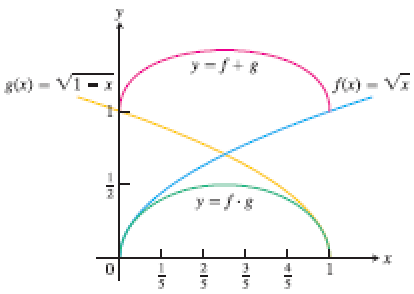
We can also define the function f/g or $\dfrac{f}{g}$ by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for } x \in D(f) \cap D(g) \text{ with } g(x) \neq 0.$$

Functions can also be multiplied by constants:

If c is a real number, the function cf is defined by $(cf)(x) = cf(x)$ for $x \in D(f)$.





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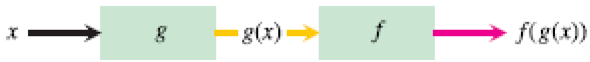
Another way of combining functions

DEFINITION. If f and g are functions, the **composite function** $f \circ g$ ("f composed with g", also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined.



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Examples:

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{3-x}$, find each function and decide on the domain.

[Note that $D(f) = [0, \infty)$ and $D(g) = (-\infty, 3]$.]

(a) $f \circ g$ (b) $g \circ f$ (c) $f \circ f$ (d) $g \circ g$

<u>Solution.</u>

(a) $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{3-x}\right) = \sqrt{\sqrt{3-x}} = \sqrt[4]{3-x}$

The domain of $f \circ g$ is $\{x \mid 3-x \geq 0\} = \{x \mid x \leq 3\} = (-\infty, 3]$

(b) $(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{3-\sqrt{x}}$

For $\sqrt{x}$ to be defined, we need $x \geq 0$. For $\sqrt{3-\sqrt{x}}$ to be defined, we need $3-\sqrt{x} \geq 0$, i.e., $\sqrt{x} \leq 3$, or $0 \leq x \leq 9$.

Thus the domain of $g \circ f$ is $[0, 9]$.

<u>**NOTE:**</u> From the above example, you can see that, in general, $f \circ g \neq g \circ f$.
Remember, the notation $f \circ g$ means that the function g is applied first and then f is applied second.

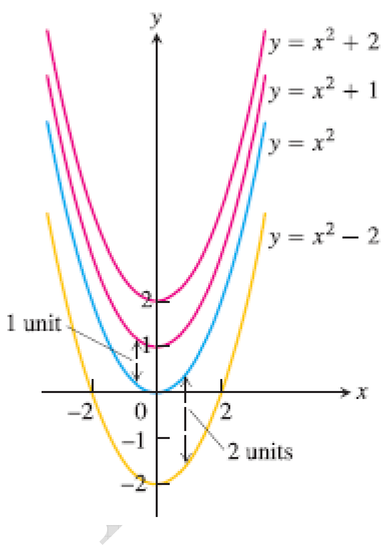
1.4 Transformations of Functions

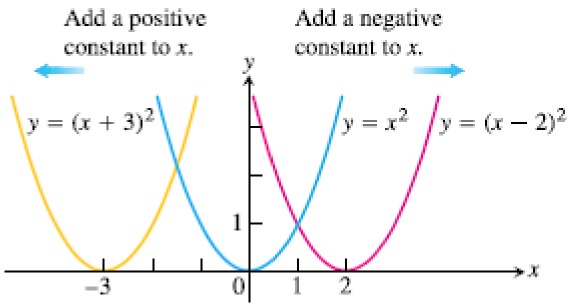
Shifting, scaling and reflecting a graph of a function

[***DO NOT memorize the following tables***. We shall discuss in class how to remember all the ideas in the following tables without memorizing. Just memorizing will not help; you will get confused. The ideas are remembered through understanding. Whenever needed, the appropriate idea will surface through understanding.]

Vertical shift	To obtain the graph of $y = f(x) + c$	Shift/translate the graph of $y = f(x)$ a distance of c units upward (Negative c would mean "$\lvert c \rvert$ units downward.")
Horizontal shift	To obtain the graph of $y = f(x + c)$	Shift/translate the graph of $y = f(x)$ a distance of c units to the left. (Negative c would mean "$\lvert c \rvert$ units to the right.")

Examples





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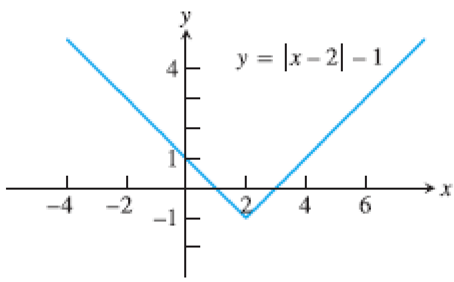
Discuss how the graph of $y = |x - 2| - 1$ can be obtained from the graph of $y = |x|$.



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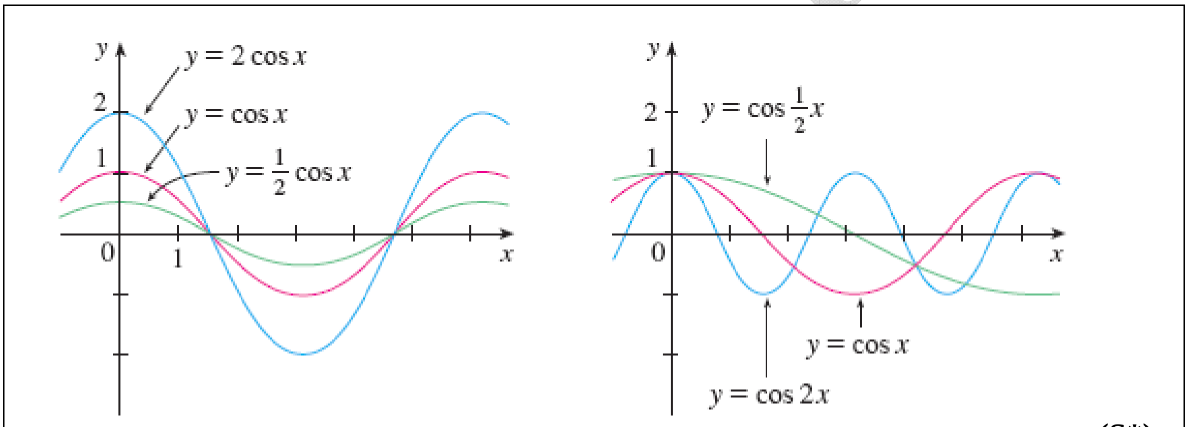
Vertical and Horizontal Scaling and Reflecting

Vertical Scaling ($c > 0$)	To obtain the graph of $y = cf(x)$	Rescale the graph of $y = f(x)$ vertically by a factor of c. ($c > 1$ would mean **stretching** while $c < 1$ would mean **shrinking**.)
Horizontal Scaling ($c > 0$)	To obtain the graph of $y = f(cx)$	Rescale the graph of $y = f(x)$ horizontally by a factor of c. ($c > 1$ would mean **shrinking** while $c < 1$ would mean **stretching**.)

Examples



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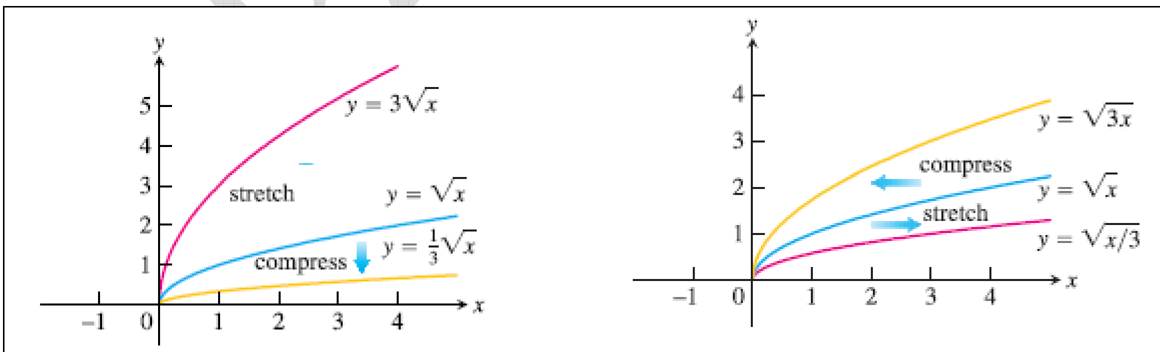

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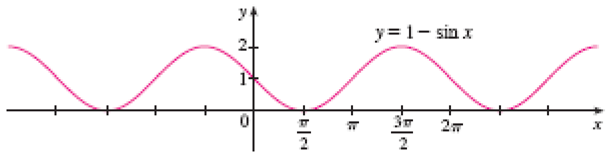
Reflecting across the *x*-axis	To obtain the graph of $y = -f(x)$	Reflect the graph of $y = f(x)$ across the *x*-axis.
Reflecting across the *y*-axis	To obtain the graph of $y = f(-x)$	Reflect the graph of $y = f(x)$ across the *y*-axis.

Example

Discuss how the graph of $y = 1 - \sin x$ can be obtained from the graph of $y = \sin x$.



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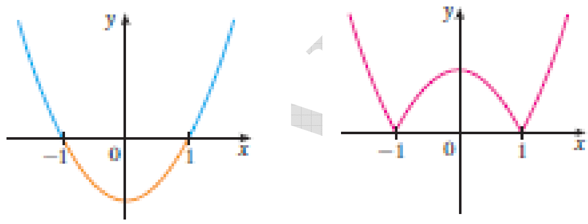
Another transformation of some interest is taking the absolute value of a function. Given the graph of $y = f(x)$, how do we obtain the graph of $y = |f(x)|$?

Recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$. Then $|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$

For the graph of $y = |f(x)|$, the part of the graph of $y = f(x)$ that lies above the *x*-axis remains the same, and the part that lies below the *x*-axis is reflect about the *x*-axis.

Sketch the graph of the function $y = |x^2 - 1|$.



(a) $y = x^2 - 1$ (b) $y = |x^2 - 1|$ (S*)

(nby, Jun 2017)